Ruprecht-Karls-Universität Heidelberg

Fakultät für Mathematik und Informatik

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A Theorem of Arnold and Jost and its application in the Kepler System

Name: Carl Walter

Matrikelnummer: 4263232

Betreuer:in: Prof. Dr. Peter Albers

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Abstract

Hamiltonian systems rarely admit explicit solutions. Integrating the equations of motion in closed form is often difficult, frequently impossible, and—when one seeks a qualitative understanding of the dynamics—not even what one is after. At the heart of this thesis lies the theorem of Arnold and Jost, which takes a different route: for completely integrable systems it produces, quite naturally and directly from the integrals of motion, a local structural picture of the solutions.

The setting is a $2n$-dimensional symplectic manifold carrying n independent integrals of motion in involution. The theorem asserts that whenever their common level set is compact and connected, it is diffeomorphic to an n-dimensional torus $\mathbb{T}^n$, and that an entire neighbourhood of this torus admits so-called *action-angle coordinates*. In these coordinates the symplectic form takes its standard Darboux shape and the Hamiltonian depends only on the action variables, so that the motion on each invariant torus is quasi-periodic.

Our exposition follows the proof of Moser and Zehnder [2005], which proceeds in three steps. We first establish the torus structure of the common level set, then construct local symplectic normal forms near each point of the torus, and finally extend these local normal forms to a coordinate system on a full neighbourhood of the torus. As an illustration of the theorem in action, the closing section treats the Kepler problem. Its rotational symmetry produces enough integrals in involution to guarantee complete integrability, and the construction of Delaunay variables makes the abstract statement of Arnold and Jost fully explicit.

Contents

1. Arnold–Jost Theorem

The necessary background on symplectic manifolds, Hamiltonian vector fields, and the Poisson bracket is collected in Appendix A; in particular, the Darboux theorem (Theorem A.1) and the notion of functions *in involution* (Definition 6) are used throughout. For a thorough treatment of symplectic geometry we refer to da Silva [2001]; for the dynamical-systems perspective underlying the proof, to Moser and Zehnder [2005].

1.1. Statement

Theorem 1.1 (Arnold–Jost). *Let (M,ω) be a $2n$-dimensional symplectic manifold and let $H : M \to \mathbb{R}$ be a Hamiltonian function. Assume that there exist n smooth functions $F_1, F_2, \ldots, F_n$ that are independent and in involution, i.e.*

$$\{F_i, F_j\} = 0 \qquad \textit{for all } i, j, \tag{1.1.1}$$

$$dF_1, \ldots, dF_n \quad \textit{are linearly independent}, \tag{1.1.2}$$

$$\{H, F_i\} = 0 \qquad \textit{for all } i. \tag{1.1.3}$$

Define

$$F := (F_1, \ldots, F_n) \in C^2(M, \mathbb{R}^n), \tag{1.1.4}$$

and suppose that $F^{-1}(0) \subset M$ is an n-dimensional submanifold, that is compact and connected.

Then $N := F^{-1}(0)$ is an embedded n-dimensional torus $\mathbb{T}^n$. Moreover, there exists an open neighbourhood $U(N) \subset M$ together with coordinates $x = (x_1, \ldots, x_n) \in \mathbb{T}^n$ and $y = (y_1, \ldots, y_n) \in D_1$ (the so-called ***action-angle coordinates****), a diffeomorphism $\psi : \mathbb{T}^n \times D_1 \to U(N)$, and a diffeomorphism $\mu : D_2 \to D_1$ with $\mu(0) = 0$, such that*

$$\mu \circ F \circ \psi = y, \tag{1.1.5}$$

$$\psi^* \omega = \sum_{j=1}^{n} dy_j \wedge dx_j, \tag{1.1.6}$$

where D_1 and D_2 are suitable domains in $\mathbb{R}^n$.

We note that the value 0 is chosen purely for convenience; the theorem applies verbatim to any other regular value of F.

1.2. Proof roadmap (three steps)

We prove Theorem 1.1 in three steps:

1. **Step I.** Show that $N = F^{-1}(0)$ is diffeomorphic to $\mathbb{T}^n$.
2. **Step II.** Construct *local* canonical coordinates near an arbitrarily chosen $p \in N$, in which the integrals only depend on one half of the coordinate chart.
3. **Step III.** Extend these local coordinates to a coordinate system on a full neighbourhood $U(N)$ of the torus.

1.3. *N is a torus*

The first step rests on the observation that the Hamiltonian vector fields associated with the F_i are tangent to N and provide a frame on N at every point. This is the content of the next lemma.

Lemma 1.2. *At every $p \in N$, the vectors $X_{F_1}(p), \dots, X_{F_n}(p)$ form a basis of T_pN.*

Proof. The map $\omega_p^\flat \colon T_pM \to T_p^*M,\ v \mapsto \omega_p(v, \cdot)$ is an isomorphism by non-degeneracy and sends each $X_{F_i}(p)$ to $-dF_i(p)$. The $dF_i(p)$ are linearly independent (regular value), hence so are the $X_{F_i}(p)$. Each lies in

$$T_pN = \ker dF_p$$

since

$$dF_j\Big(X_{F_i}\Big) = -\,\omega(X_{F_j}, X_{F_i}) = \{F_j, F_i\} = 0,$$

and $\dim T_pN = n$ forces the n independent vectors to span. Q.E.D.

With this lemma at our disposal we can now carry out Step I of the theorem.

Proof of Theorem 1.1 (Step 1: N is a torus). Let $\phi_i^{t_i}$ denote the flow of the Hamiltonian vector field X_{F_i}. The flows commute, $\phi_i^{t_i} \circ \phi_j^{t_j} = \phi_j^{t_j} \circ \phi_i^{t_i}$, because the involution of the integrals implies

$$[X_{F_i}, X_{F_j}] = X_{\{F_i, F_j\}} = X_0 = 0,$$

and the commutator of two vector fields vanishes precisely when their flows commute.

Define the composition of these flows as $\phi^t := \phi_1^{t_1} \circ \cdots \circ \phi_n^{t_n}$ for $t = (t_1, \dots, t_n) \in \mathbb{R}^n$. Clearly,

$$\phi^0 = \phi_1^0 \circ \cdots \circ \phi_n^0 = id \circ \cdots \circ id$$

and as the individual flows commute, we also immediately find

$$\phi^{t+s} = \phi^t \circ \phi^s.$$

It remains to show that this joint flow is defined for every $t \in \mathbb{R}^n$.

Global existence. Since N is compact and each vector field X_{F_i} is smooth and tangent to N, its restriction $X_{F_i}|_N$ is a smooth vector field on the compact submanifold N. By the Picard–Lindelöf theorem, the flow $\phi_i^{t_i}$ therefore exists for every $t_i \in \mathbb{R}$, and $\phi^t(p)$ is well defined for all $p \in N$ and all $t \in \mathbb{R}^n$.

Thus, ϕ gives a smooth action of $\mathbb{R}^n$ on N via

$$t \mapsto \phi^t(p), \qquad p \in N. \tag{1.3.1}$$

Fix a point $p \in N$ and consider the associated orbit map

$$\phi_p \colon \mathbb{R}^n \to N, \qquad \phi_p(t) := \phi^t(p). \tag{1.3.2}$$

We first show that ϕ_p is a submersion at every $t \in \mathbb{R}^n$. For each $i = 1, \dots, n$, the chain rule and the defining property of a flow give

$$d(\phi_p)_t\left(\frac{\partial}{\partial t_i}\right) = \frac{\partial}{\partial t_i}\phi^t(p) = X_{F_i}(\phi^t(p)), \tag{1.3.3}$$

where we have used the commutativity of the flows $\phi_1, \dots, \phi_n$ to move the i-th flow to the outside. By Lemma 1.2, the vectors $X_{F_1}(\phi^t(p)), \dots, X_{F_n}(\phi^t(p))$ form a basis of $T_{\phi^t(p)}N$. Hence $d(\phi_p)_t$ is an isomorphism for every t, so ϕ_p is both a submersion and an immersion.

Hence, ϕ_p is a local diffeomorphism: every $t \in \mathbb{R}^n$ admits an open neighbourhood U on which $\phi_p|_U$ is a diffeomorphism onto its open image $\phi_p(U)$. Consequently $\mathrm{im}(\phi_p)$ is open in N.

The $\mathbb{R}^n$-action partitions N into disjoint orbits, and we have just seen that every orbit is open. The complement of $\mathrm{im}(\phi_p)$ is itself a union of orbits, hence open; so $\mathrm{im}(\phi_p)$ is both open and closed in N. Since N is connected we conclude

$$\mathrm{im}(\phi_p) = N, \tag{1.3.4}$$

i.e. ϕ_p is surjective.

It remains to identify the kernel of the action. Consider the isotropy subgroup of p,

$$\Gamma := \{\, t \in \mathbb{R}^n \mid \phi^t(p) = p \,\}. \tag{1.3.5}$$

The relations $\phi^{t+s} = \phi^t \circ \phi^s$ and $p = \phi^{-t} \circ \phi^t(p)$ show that Γ is closed under addition and inversion, so it is a subgroup of $\mathbb{R}^n$:

$$\phi^{t+s} = \phi^t \circ \phi^s, \qquad \phi^{-t}(p) = p \text{ for } s, t \in \Gamma. \tag{1.3.6}$$

We next claim that Γ is *discrete*, i.e. that some neighbourhood U of $0 \in \mathbb{R}^n$ satisfies $U \cap \Gamma = \{0\}$.
Since ϕ_p is a local diffeomorphism, we may choose U small enough that $\phi_p|_U$ is injective; for $t \in \Gamma \cap U$ we then have

$$\phi_p(t) = p = \phi_p(0) \implies t = 0. \tag{1.3.7}$$

A standard result from linear algebra now identifies discrete subgroups of $\mathbb{R}^n$ with lattices; we record it here without proof:

Lemma 1.3. *A subset $\Lambda \subset \mathbb{R}^n$ is a lattice if and only if it is a discrete subgroup of $\mathbb{R}^n$.*

Applying this to Γ, we obtain vectors $v_1, \dots, v_d \in \mathbb{R}^n$ such that every $\gamma \in \Gamma$ can be written as $\gamma = \sum_{k=1}^{d} g_k v_k$ for some $g_k \in \mathbb{Z}$.

Since ϕ_p is a surjective local diffeomorphism whose fibres are precisely the cosets of Γ, it descends to a diffeomorphism

$$\mathbb{R}^n/\Gamma \;\cong\; N. \tag{1.3.8}$$

As N is compact, the quotient $\mathbb{R}^n/\Gamma$ is also compact, which forces Γ to be a lattice of full rank n. Consequently

$$\Gamma \cong \mathbb{Z}^n, \tag{1.3.9}$$

and therefore

$$N \cong \mathbb{R}^n/\mathbb{Z}^n = \mathbb{T}^n, \tag{1.3.10}$$

which completes Step I. Q.E.D.

1.4. Local action-angle coordinates near points of N

In Step II we introduce the coordinates x and y of Theorem 1.1 locally near a point $p \in N$. The construction is an application of Darboux's theorem to a slightly richer setting, in which the F_i themselves play the role of half of the coordinate functions.

Theorem 1.4 (An application of Darboux's theorem)**.** *Let (M, ω) be a symplectic manifold. If $F_1, \ldots, F_n$ are n functions satisfying (1.1.1) and (1.1.2), then every point $p \in M$ admits a symplectic coordinate chart $\psi : V \to U$, where $U \subset M$ is a neighbourhood of p and $V \subset \mathbb{R}^{2n}$ is a neighbourhood of the origin, such that*

$$\psi^*\omega = \sum_{j=1}^{n} dy_j \wedge dx_j, \qquad F_j \circ \psi = y_j, \tag{1.4.1}$$

where (x, y) are the standard coordinates on $\mathbb{R}^{2n}$.

The theorem in fact holds for any number $s \leq n$ of commuting integrals; for our purposes only the case $s = n$ is required.

Proof of Theorem 1.4 in the case $s = n$. By Darboux's theorem (A.1) we obtain canonical coordinates (U, ϕ), i.e.

$$\phi^*\omega = \omega = \sum_{k=1}^{n} dy_k \wedge dx_k. \tag{1.4.2}$$

and U being a neighborhood of arbitrary p.

We denote $f_j := F_j \circ \phi$ and show, that the f_j are in involution.

$$\begin{aligned} \{f_j, f_k\} = \{F_j \circ \phi, F_k \circ \phi\} = d(F_j \circ \phi)(X_{F_k \circ \phi}) = dF_j \circ (\phi' \circ (\phi')^{-1})(X_{F_k}) \\ = dF_j(X_{F_k}) = \{F_j, F_k\} = 0 \end{aligned} \tag{1.4.3}$$

and the df_j are also linearly independent as ϕ is a diffeomorphism. By a transformational law, proven in Moser and Zehnder [2005, Chapter I, §1.4], we can canonically transform this by χ, such that

$$f_j \circ \chi = y_j \tag{1.4.4}$$

$$\chi^*\omega = \omega \Rightarrow (\phi \circ \chi)^*\omega = \omega \tag{1.4.5}$$

Thus $\psi := \phi \circ \chi$ concludes the proof.

Q.E.D.

Two coordinate systems both satisfying (1.4.1) need not coincide, but their difference is severely constrained. The next lemma pins down the precise form of this freedom.

Lemma 1.5 (Addition to Theorem 1.4)**.** *Let* (x, y) *and* (x', y') *be coordinates both fulfilling* (1.4.1)*. Then there exists a smooth function* $Q : \mathbb{R}^n \to \mathbb{R}$ *such that*

$$x' = x + \frac{\partial Q}{\partial y}, \qquad y' = y. \tag{1.4.6}$$

Proof. Step 1: $y' = y$*.* The defining identity $F_j \circ \psi = y_j$ holds for both charts, hence $y'_j = F_j \circ \psi = y_j$.

Step 2: $x' = x + \partial Q/\partial y$*.* Using the previous step,

$$\sum_{j=1}^{n} dy'_j \wedge dx'_j = \sum_{j=1}^{n} dy_j \wedge dx'_j = \psi^* \omega = \sum_{j=1}^{n} dy_j \wedge dx_j, \tag{1.4.7}$$

from which $dx_j = dx'_j$ for every j. The differences $x'_j - x_j$ therefore depend on y alone, so $x'_j = x_j + f_j(y)$ for some smooth f_j. Since $\sum_j f_j(y)\, dy_j$ is closed (which follows because $\sum_j df_j(y) \wedge dx_j = 0$ identically), there exists a potential Q with $\partial Q/\partial y_j = f_j(y)$. Q.E.D.

1.5. Establishment of the final coordinates

The remaining task is to extend the local coordinates generated in Step II to a full neighbourhood $U(N)$ of the torus. Throughout this section we fix a point $p \in N$ and denote the chart of Section 1.4 by

$$\psi : V \to U, \tag{1.5.1}$$

where $V = V_1 \times V_2 \subset \mathbb{R}^{2n}$ is an open neighbourhood of the origin and $U \subset M$ is a neighbourhood of p.

The key observation is that, in the chart ψ, the Hamiltonian flows $\phi_j^{t_j}$ become coordinate translations.

Claim 1.6. *The flow $\phi_j^{t_j}$ acts on the chart ψ by translation by t_j in the x_j-direction, i.e. in local coordinates $(x, y) \mapsto (x + t_j e_j, y)$ for t_j sufficiently small.*

To prove this we use a standard result from symplectic geometry:
Lemma 1.7. *For a symplectomorphism $g : (\mathbb{R}^{2n}, \omega_0) \to (M, \omega)$ (that is, a diffeomorphism with $g^*\omega = \omega_0$) and any smooth function f on M,*

$$g_*(X_{f\circ g}) = X_f. \tag{1.5.2}$$

Proof of Claim 1.6. Both sides of the identity

$$\phi_j^{t_j} \circ \psi(x, y) = \psi(x + t_j e_j, y) \tag{1.5.3}$$

are smooth curves in M that agree at $t_j = 0$ (both equal $\psi(x, y)$). By the uniqueness theorem for ODEs (Picard–Lindelöf), two integral curves of the same vector field with the same initial condition coincide. It therefore suffices to verify that the two curves satisfy the same ODE in t_j, i.e. that their derivatives agree:

$$\frac{d}{dt_j}\Big[\phi_j^{t_j} \circ \psi(x, y)\Big] = X_{F_j}(\phi_j^{t_j} \circ \psi(x, y)), \qquad \frac{d}{dt_j}\Big[\psi(x + t_j e_j, y)\Big] = D\psi(x + t_j e_j, y) \cdot e_j. \tag{1.5.4}$$

The left-hand identity is just the definition of the flow $\phi_j^{t_j}$. For the right-hand side, observe that $\frac{\partial}{\partial x_j} = X_{y_j} = X_{F_j \circ \psi}$, since

$$\omega\left(\frac{\partial}{\partial x_j}(p),\, v\right) = \sum_{k=1}^{n} dy_k \wedge dx_k \left(\frac{\partial}{\partial x_j}(p),\, v\right) = dy_j(v).$$

Applying Lemma 1.7 with $f = F_j$ and $g = \psi$ then gives

$$D\psi \circ X_{F_j \circ \psi} = X_{F_j}(\psi(\cdot)),$$

so the two curves satisfy the same ODE. By Picard–Lindelöf they coincide, which proves the claim. Q.E.D.

Since Claim 1.6 holds for every j, we are motivated to define the composite map

$$\theta(x, y) := \phi^x \circ \psi(0, y),$$

which extends ψ in the x-direction. We need to specify the possible domains of θ to use it as we progress with the proof and claim:

Claim 1.8. *(Extension of the domain of θ.) For every compact set $K \subset \mathbb{R}^n$ there exists a neighbourhood $D_2(K)$ of $0 \in \mathbb{R}^n$ such that θ is defined for all $(x, y) \in K \times D_2(K)$.*

Proof. We argue inductively, starting with $n = 1$. Consider the orbit $\phi^t(\psi(0, 0))$. It is defined for every $t \in \mathbb{R}$ because $\psi(0, 0) = p \in N$ and the flow on N exists globally; moreover it is smooth, and in particular continuously differentiable. The local version of Picard–Lindelöf therefore applies and yields

$$\forall p \in M \;\; \exists \, \text{neighbourhood } U \;\; \exists \, I_\epsilon \ni 0 \,:\; \phi^t(\psi(q)) \text{ is defined } \forall q \in U,\, t \in I_\epsilon. \tag{1.5.5}$$

Using the flow property $\phi^{t+s}(p) = \phi^t(\phi^s(p))$, we obtain analogously that for every $t \in K$ there exist $\epsilon^{(t)} > 0$ and a neighbourhood U_t of p such that

$$(t - \epsilon^{(t)}, t + \epsilon^{(t)}) \times U_t \to M, \quad (s, q) \mapsto \phi^s(q) \tag{1.5.6}$$

is well defined.

Compactness of K now produces a finite subcover: there exist points $t_1, \ldots, t_k \in K$ with $K \subset \bigcup_{i=1}^k (t_i - \epsilon^{(t_i)}, t_i + \epsilon^{(t_i)})$. The intersection $U := \bigcap_{i=1}^k U_{t_i}$ is open (the family is finite), and the flow is defined on $K \times U$. The case $n > 1$ follows by induction on the coordinate directions, using the topological fact that the image of a compact set under a continuous function is compact. Q.E.D.

From now on we view $\theta : K \times D_2(K) \to M$ as defined on a compact box $K \subset \mathbb{R}^n$ containing every interval $(-v_i, v_i)$, where the v_i span the lattice Γ from (1.3.5). Our next aim is to show that θ is symplectic.

Lemma 1.9. *The map $\theta : K \times D_2(K) \to M$ given by $\theta(x, y) = \phi^x(\psi(0, y))$ satisfies*

$$\theta^*\omega = \omega_0. \tag{1.5.7}$$

Proof. Define θ_{-s} by $\theta_{-s}(x, y) = \theta(x - s, y) = \phi^{-s} \circ \theta(x, y)$. Then

$$\theta_{-s}^*\omega = (\phi^{-s} \circ \theta)^*\omega = \theta^*\Big((\phi^{-s})^*\omega\Big) = \theta^*\omega, \tag{1.5.8}$$

where the last step uses $(\phi^{-s})^*\omega = \omega$. This identity in turn holds because Hamiltonian flows are symplectomorphisms: indeed

$$\mathcal{L}_{X_{F_i}}\omega = d(\iota_{X_{F_i}}\omega) + \iota_{X_{F_i}}d\omega = d(-dF_i) + 0 = 0.$$

Hence $\theta_{-s}^*\omega$ is independent of s. Evaluating at $s = x$ gives $\theta_{-x}^*\omega = \theta_0^*\omega$; but $\theta_0(x, y) = \theta(0, y) = \psi(0, y)$, and for x near 0 we have $\theta = \psi$, so $\psi^*\omega = \omega_0$ by construction. Translating back yields $\theta^*\omega = \omega_0$. Q.E.D.

To obtain a lattice structure for the preimages, we next identify which pairs (x, y) map to the same point in M under θ (compare to 1.3). For y near 0 we seek $w_k(y) \in \mathbb{R}^n$ satisfying

$$\theta(w_k(y), y) = \theta(0, y), \qquad w_k(0) = v_k, \tag{1.5.9}$$

where $v_1, \ldots, v_n$ are once again the basis vectors of the lattice Γ from Section 1.3. The initial condition $w_k(0) = v_k$ is natural, since

$$\theta(v_k, 0) = \phi^{v_k} \circ \psi(0, 0) = \psi(0, 0) = \theta(0, 0). \tag{1.5.10}$$

To solve (1.5.9) we introduce the auxiliary map

$$\rho : (\xi, \eta) \mapsto \psi^{-1} \circ \theta(v_k + \xi, \eta), \tag{1.5.11}$$

which is well defined for (ξ, η) near $(0, 0)$ – precisely where θ agrees with ψ – and which satisfies $\rho(0, 0) = (0, 0)$. Equivalently, $\rho = \psi^{-1} \circ \theta_{v_k}$.

Since $F_j \circ \psi = y_j$ is an integral by construction (Theorem 1.4), we obtain

$$y_j = F_j \circ \psi(x, y) = F_j \circ \psi(\rho(\xi, \eta)) = F_j \circ \theta(v_k + \xi, \eta) = F_j \circ \phi^{v_k} \circ \psi(\xi, \eta) = F_j \circ \psi(\xi, \eta) = \eta_j. \tag{1.5.12}$$

Combined with (1.5.8), this gives

$$\theta^* \omega = \sum_{i=1}^{n} dy_i \wedge dx_i, \qquad \theta_{v_k}^* \omega = \sum_{i=1}^{n} d\eta_i \wedge d\xi_i, \tag{1.5.13}$$

and therefore

$$\rho^* \left(\sum_{i=1}^{n} dy_i \wedge dx_i \right) = \sum_{i=1}^{n} d\eta_i \wedge d\xi_i. \tag{1.5.14}$$

So ρ satisfies the hypotheses of Theorem 1.4, and by Lemma 1.5 there exists a smooth function Q_k with

$$(\xi, \eta) \mapsto \left(x = \xi + \frac{\partial Q_k}{\partial \eta}(\eta),\ y = \eta \right), \qquad \frac{\partial Q_k}{\partial \eta}(0) = 0. \tag{1.5.15}$$

We can now solve (1.5.9) by setting $w_k(y) = v_k + \xi$ and reading off ξ from the equation $\theta(\xi + v_k, y) = \theta(0, y) = \psi(0, y)$ for y near 0. By the analysis above,

$$\theta(\xi + v_k, y) = \psi \circ \rho(\xi, y) = \psi \left(\xi + \frac{\partial Q_k}{\partial y}(y), y \right), \tag{1.5.16}$$

so the equation reduces to

$$\psi(0, y) = \psi \left(\xi + \frac{\partial Q_k}{\partial y}(y), y \right). \tag{1.5.17}$$

Since ψ is a diffeomorphism, hence injective, this has the unique solution

$$\xi = -\frac{\partial Q_k}{\partial y}(y), \tag{1.5.18}$$

so $w_k(y) = v_k - \partial Q_k / \partial y = \partial W_k / \partial y$ for

$$W_k(y) := \langle v_k, y \rangle - Q_k(y). \tag{1.5.19}$$

We have thus established the existence of the w_k together with their smooth dependence on y.

A short determinant argument shows that the $w_k(y)$ remain linearly independent for y in a sufficiently small neighbourhood of 0. Consider

$$A(y) := \Big(w_1(y), \dots, w_n(y)\Big) \in \mathbb{R}^{n\times n}, \qquad a(y) := \det A(y). \tag{1.5.20}$$

The function a is smooth in y, and at $y = 0$ the columns of $A(0)$ coincide with the lattice basis $v_1, \dots, v_n$, so $a(0) \neq 0$. By continuity, $a(y) \neq 0$ on a neighbourhood of 0, which establishes the desired linear independence.

The relation

$$\theta(x + w_k(y), y) = \phi^{x+w_k(y)} \circ \theta(0, y) = \phi^x \circ \theta(0, y) = \phi^x \circ \psi(0, y) = \theta(x, y) \tag{1.5.21}$$

now permits us to extend θ to a map $\theta : \mathbb{R}^n \times D_2 \to M$. The extension again satisfies the conclusion of Lemma 1.4, and we let $\Gamma(y)$ denote the lattice spanned by the vectors $w_k(y)$. We then obtain a diffeomorphism

$$\theta_0 : \mathbb{R}^n/\Gamma(y) \to F^{-1}(y) \cap U(N), \qquad [x] \mapsto \theta(x, y), \tag{1.5.22}$$

where $U(N)$ is a neighbourhood of $N = F^{-1}(0)$ (continuity of F^{-1} ensures that θ_0 is well defined). It remains to verify that θ_0 is injective for y in some smaller neighbourhood of 0.

Proof that θ_0 is injective. Assume for contradiction that no such neighbourhood exists. Then there is a sequence $y_n \to 0$ together with two sequences $x_n, \tilde{x}_n$ satisfying

$$\theta(x_n, y_n) = \theta(\tilde{x}_n, y_n) \qquad \text{for all } n, \tag{1.5.23}$$

and with $x_n - \tilde{x}_n \notin \Gamma(y_n)$. Passing to a subsequence we may assume $x_n \to x^*$ and $\tilde{x}_n \to \tilde{x}^*$.

By continuity of θ we then have $\theta(x^*, 0) = \theta(\tilde{x}^*, 0)$, i.e. $x^* - \tilde{x}^* \in \Gamma(0) = \Gamma$. Indeed, $w_k(0) = v_k$ and

$$\begin{aligned} \phi^{x^*}(p) = \theta(x^*, 0) = \theta(\tilde{x}^*, 0) = \phi^{\tilde{x}^*}(p) \\ \iff \phi^{x^* - \tilde{x}^*}(p) = p, \end{aligned} \tag{1.5.24}$$

recalling that $\psi(0, 0) = p$.

But this implies that $x_n - \tilde{x}_n$ approaches a point of Γ as $n \to \infty$: for every $\epsilon > 0$ there exists an index N such that the distance from $x_n - \tilde{x}_n$ to some point of $\Gamma(y_n)$ is less than ϵ whenever $n > N$. Since $\Gamma(y_n)$ depends continuously on y_n and each $\Gamma(y_n)$ is discrete, the spacing between lattice points is bounded below by a uniform constant $\delta > 0$. Consequently $x_n - \tilde{x}_n \in \Gamma(y_n)$ for n sufficiently large, contradicting our assumption. Q.E.D.

Having obtained a diffeomorphism, it remains only to normalise the periods of x, i.e. to straighten the lattice $\Gamma(y)$ to the standard integer lattice $\mathbb{Z}^n$ uniformly in y. To carry this out cleanly (preserving canonicity) we introduce a tool that is used frequently in symplectic geometry: *generating functions* of canonical transformations.

Generating functions of canonical transformations

In the standard local model $(\mathbb{R}^{2n}, \omega_0)$ with coordinates (x, y) and $\omega_0 = \sum_j dy_j \wedge dx_j$, a Type-2 generating function is a smooth function

$$S \;=\; S(x',\, y) \colon U \subset \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R} \tag{1.5.25}$$

satisfying the non-degeneracy condition

$$\det\left(\frac{\partial^2 S}{\partial x'_k\, \partial y_j}\right)_{j,k} \;\neq\; 0 \qquad \text{on } U. \tag{1.5.26}$$

Given such an S, the implicit relations

$$y'_k \;=\; \frac{\partial S}{\partial x'_k}(x', y), \qquad x_j \;=\; \frac{\partial S}{\partial y_j}(x', y), \tag{1.5.27}$$

define a local diffeomorphism $\sigma \colon (x', y') \mapsto (x, y)$ on a neighbourhood of any point where (1.5.26) holds. Notice how the non-degeneracy of the Hesse matrix of S is required for enabling us to implicitly define σ by the Inverse Function Theorem. The following is the standard fact we will rely on; for a complete proof see Moser and Zehnder [2005, Chapter I, §1.4].

Lemma 1.10. *Every diffeomorphism σ defined by a Type-2 generating function via (1.5.27) is canonical, i.e.*

$$\sigma^*\Big(\textstyle\sum_{j=1}^n dy_j \wedge dx_j\Big) \;=\; \sum_{k=1}^n dy'_k \wedge dx'_k. \tag{1.5.28}$$

The proof is a short direct calculation in differentials, included at the end of this subsection for completeness.

The lattice-straightening generating function

Recall the smooth potentials W_k from (1.5.19), whose gradients $\partial W_k/\partial y$ generate the lattice $\Gamma(y)$ at each y. We now take

$$S(x', y) \ := \ \sum_{k=1}^{n} x'_k \, W_k(y). \tag{1.5.29}$$

The generating-function recipe (1.5.27) yields the explicit relations

$$y'_k \ = \ \frac{\partial S}{\partial x'_k} \ = \ W_k(y), \qquad x_j \ = \ \frac{\partial S}{\partial y_j} \ = \ \sum_{k=1}^{n} x'_k \, \frac{\partial W_k}{\partial y_j}(y). \tag{1.5.30}$$

This is exactly the lattice-straightening transformation we wanted: y'_k records the lattice spanned at y, while the x_j are linear in x' with coefficients $\partial W_k/\partial y_j(y)$. The non-degeneracy condition (1.5.26) reads

$$\det\!\left(\frac{\partial^2 S}{\partial x'_k \, \partial y_j}\right) \ = \ \det\!\left(\frac{\partial W_k}{\partial y_j}\right) \ = \ \det A(y) \ = \ a(y) \ \neq \ 0 \tag{1.5.31}$$

for y in a neighbourhood of 0, by the determinant argument given just before (1.5.22). So σ is well defined on such a neighbourhood, and by Lemma 1.10 it is automatically canonical:

$$\sigma^*\Big(\textstyle\sum_{j=1}^{n} dy_j \wedge dx_j\Big) \ = \ \displaystyle\sum_{k=1}^{n} dy'_k \wedge dx'_k. \tag{1.5.32}$$

Periodicity and conclusion

The first relation in (1.5.30) shows that, at fixed y' (equivalently fixed y), the unit shift $x' \mapsto x' + e_\ell$ corresponds to $x \mapsto x + w_\ell(y)$, i.e. to translation by the ℓ-th generator of $\Gamma(y)$. Hence integer translations in x' correspond bijectively to lattice translations in x, and the quotient map $\mathbb{R}^n \to \mathbb{R}^n/\Gamma(y)$ becomes, in the x'-coordinates, the standard projection $\mathbb{R}^n \to \mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n$.

Setting $\Psi := \theta \circ \sigma$ (the final action-angle chart, to distinguish it from the local Darboux chart ψ of Section 1.4), the symplectic form pulls back correctly:

$$\Psi^*\omega \ = \ \sigma^*\theta^*\omega \ = \ \sigma^*\Big(\textstyle\sum_j dy_j \wedge dx_j\Big) \ = \ \displaystyle\sum_k dy'_k \wedge dx'_k, \tag{1.5.33}$$

by Lemma 1.9 and (1.5.32). With $\mu := W^{-1}\colon D_2 \to D_1$ – a diffeomorphism by the determinant argument referred to above, with $\mu(0) = 0$ – the identity $\mu \circ F \circ \Psi = y'$ in (1.1.5) follows from $F \circ \theta = y$, which itself follows from $F_j \circ \psi = y_j$ in Theorem 1.4 together with the construction $\theta(x, y) = \phi^x \circ \psi(0, y)$ and the invariance of each F_j along its own flow. This completes the proof of Theorem 1.1.

Proof of Lemma 1.10.

Differentiating the implicit relations (1.5.27),

$$dy'_k = \sum_j \frac{\partial^2 S}{\partial x'_k\, \partial y_j}\, dy_j + \sum_l \frac{\partial^2 S}{\partial x'_k\, \partial x'_l}\, dx'_l, \tag{1.5.34}$$

$$dx_j = \sum_k \frac{\partial^2 S}{\partial y_j\, \partial x'_k}\, dx'_k + \sum_l \frac{\partial^2 S}{\partial y_j\, \partial y_l}\, dy_l. \tag{1.5.35}$$

Multiplying these two and summing over the appropriate indices produces four terms; we collect them by type. The cross term $dy'_k \wedge dx'_l$ has coefficient

$$-\sum_j \frac{\partial^2 S}{\partial x'_k\, \partial y_j} \frac{\partial^2 S}{\partial y_j\, \partial x'_l}, \tag{1.5.36}$$

which is symmetric in (k, l) by Schwarz's theorem, hence vanishes when wedged into the antisymmetric $dx'_k \wedge dx'_l$. The pure $dy_j \wedge dy_l$- and $dx'_k \wedge dx'_l$-terms vanish for the same reason. What remains is exactly

$$\sum_j dy_j \wedge dx_j \;=\; \sum_k dy'_k \wedge dx'_k, \tag{1.5.37}$$

which is (1.5.28). Q.E.D.

In the remainder of the thesis we apply Theorem 1.1 to one of the most classical integrable systems – the Kepler problem – and recover the action–angle variables introduced by Delaunay [Arnold, 1989].

2. Delaunay variables

The Delaunay variables, introduced by Charles-Eugène Delaunay in the nineteenth century in his monumental study of lunar motion, are the classical action-angle variables of the Kepler problem. They have since become a cornerstone of celestial mechanics: they provide a transparent description of Keplerian orbits and serve as the natural starting point for perturbation theory in the two-body and restricted three-body problems [Arnold, 1989].

2.1. The Kepler system and its integrals

We introduce action-angle variables – the Delaunay variables – for the Kepler problem in $\mathbb{R}^n$. Its Hamiltonian is

$$H = \tfrac{1}{2}|p|^2 - |q|^{-1}. \tag{2.1.1}$$

A glance at the Hamiltonian equations

$$\dot{q}_i = \frac{\partial H}{\partial p_i} = p_i, \qquad \dot{p}_i = -\frac{\partial H}{\partial q_i} = \frac{q_i}{|q|^3}, \tag{2.1.2}$$

shows that the system is invariant under rotations of configuration space. This invariance suggests that the components of angular momentum should be conserved. Concretely, we define

$$\Gamma_{ij} = q_i p_j - q_j p_i, \qquad i, j = 1, \dots, n, \tag{2.1.3}$$

and verify by direct computation that they are integrals of motion:

$$\{H, \Gamma_{ij}\} = \omega(X_H, X_{\Gamma_{ij}}). \tag{2.1.4}$$

Hamilton's equations (Definition 5) give

$$\begin{aligned} X_{\Gamma_{ij}} &= p_i \frac{\partial}{\partial p_j} + q_i \frac{\partial}{\partial q_j} - p_j \frac{\partial}{\partial p_i} - q_j \frac{\partial}{\partial q_i}, \\ \omega(X_H, X_{\Gamma_{ij}}) &= -dH(X_{\Gamma_{ij}}) = X_{\Gamma_{ij}}(H) \\ &= p_i p_j + \frac{q_i q_j}{|q|^3} - p_j p_i - \frac{q_j q_i}{|q|^3} = 0. \end{aligned} \tag{2.1.5}$$

So the Γ_{ij} are conserved. They are however *not* in involution with each other, so we cannot apply Arnold–Jost to them directly. Their Poisson brackets form an $\mathfrak{so}(n)$ Lie algebra; a direct computation (carried out in Appendix B.1) yields the formula

$$\{\Gamma_{ij}, \Gamma_{\alpha\beta}\} = \Gamma_{j\alpha}\delta_{i\beta} - \Gamma_{i\alpha}\delta_{j\beta} - \Gamma_{j\beta}\delta_{i\alpha} + \Gamma_{i\beta}\delta_{j\alpha}. \tag{2.1.6}$$

The natural next question is whether one can extract from the Γ_{ij} a maximal family of functions that *are* in involution. The classical answer is provided by the following partial Casimirs: for $k = 2, \ldots, n$ set

$$G_k^2 := \sum_{1\leq i<j\leq k} \Gamma_{ij}^2. \tag{2.1.7}$$

Thus $G_2^2 = \Gamma_{12}^2$, $G_3^2 = \Gamma_{12}^2 + \Gamma_{13}^2 + \Gamma_{23}^2$, and so on. Geometrically, G_k is the magnitude of the angular momentum restricted to the first k coordinate directions.

Lemma 2.1. *The functions $G_2^2, \ldots, G_n^2$ are in involution: $\{G_k^2, G_\ell^2\} = 0$ for all k, ℓ.*

The proof is a direct calculation utilizing Leibniz-Rule and the symmetry of the Poisson-bracket; the complete computation is given in Appendix B.2.

With involution established, the next step towards applying Theorem 1.1 to the Kepler problem is the linear independence of the differentials dG_j on a suitable domain. This is the subject of the following subsection.

2.2. Flows of the integrals and their orbits

We first establish the periodicity of the flows generated by the vector fields $X_{G_j^2}$.

Since the G_j^2 all behave analogously to G_n^2 – they are simply G_n^2 restricted to the subspace $\mathbb{R}^{2j}$ – it is sufficient to only study the flow of G_n^2. A short computation shows, that

$$F := G_n^2 = \sum_{1\le i<j\le n} (q_i p_j - q_j p_i)^2 = |q|^2|p|^2 - \langle q,p\rangle^2 = \det(S(q,p)), \tag{2.2.1}$$

where

$$S(q,p) \ := \ \begin{pmatrix} \langle q,q\rangle & \langle q,p\rangle \\ \langle p,q\rangle & \langle p,p\rangle \end{pmatrix} \in \mathrm{Sym}^+(2,\mathbb{R}) \tag{2.2.2}$$

is the 2×2 Gram matrix of the pair (q,p).

We will exploit the symmetry of S under multiplication with matrices of $SL(2,\mathbb{R})$, which is a Lie group.

Why $SL(2,\mathbb{R})$? The matrix $S(q,p)$ is built from the inner products of q and p, so it knows about the two-plane spanned by them, not about their individual lengths. Any change of basis in this two-plane that has determinant one will leave $\det S$ unchanged; rescalings of the basis with determinant $\neq 1$ will multiply $\det S$ by the square of that determinant. This suggests testing the linear group $SL(2,\mathbb{R}) = \{g \in \mathbb{R}^{2\times 2} : \det g = 1\}$ as a candidate symmetry, with elements acting on the pair (q,p) as 2×2 matrices on a column vector with values in $\mathbb{R}^n$.

The action. For $g = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in SL(2,\mathbb{R})$, define

$$\psi^g(q,p) \ := \ (a\,q + b\,p,\ c\,q + d\,p). \tag{2.2.3}$$

The group property $\psi^{gh} = \psi^g \circ \psi^h$ is immediate from matrix multiplication, and $\psi^e(q,p) = (q,p)$ for $g = I_2$. So ψ is a smooth left action of $SL(2,\mathbb{R})$ on $\mathbb{R}^{2n}$.

The Hamiltonian H is invariant. The key observation is that S transforms by congruence under ψ^g:

$$S\big(\psi^g(x,y)\big) \ = \ g\,S(x,y)\,g^T. \tag{2.2.4}$$

Indeed, the matrix $S(x,y)$ is the product $\left(\begin{smallmatrix} x^T \\ y^T \end{smallmatrix}\right)\big(x\ y\big)$ (where x, y are treated as column vectors in $\mathbb{R}^n$). Under ψ^g,

$$\begin{pmatrix} x \\ y \end{pmatrix} \;\mapsto\; g\begin{pmatrix} x \\ y \end{pmatrix}, \qquad (x\ y) \;\mapsto\; (x\ y)\, g^T, \tag{2.2.5}$$

so S transforms as gSg^T. Taking determinants,

$$\det S\Big(\psi^g(x,y)\Big) \;=\; \det\Big(g\, S\, g^T\Big) \;=\; (\det g)^2\, \det S \;=\; \det S(x,y). \tag{2.2.6}$$

Hence $H \circ \psi^g = H$ for every $g \in SL(2,\mathbb{R})$.

Linearity of ψ^g and $\det g = 1$ give the canonicity of the action directly. Writing $\psi^g(q,p) = (q', p')$ with

$$q' \;=\; a\, q + b\, p, \qquad p' \;=\; c\, q + d\, p, \tag{2.2.7}$$

we have

$$dq_i' \;=\; a\, dq_i + b\, dp_i, \qquad dp_i' \;=\; c\, dq_i + d\, dp_i, \tag{2.2.8}$$

so

$$\sum_{i=1}^n dp_i' \wedge dq_i' \;=\; \sum_{i=1}^n \Big(c\, dq_i + d\, dp_i\Big) \wedge \Big(a\, dq_i + b\, dp_i\Big) \;=\; (ad - bc) \sum_{i=1}^n dp_i \wedge dq_i \;=\; \det(g)\, \omega_0 \;=\; \omega_0, \tag{2.2.9}$$

using $\det g = 1$ in the last step. Hence

$$(\psi^g)^* \omega_0 \;=\; \omega_0, \tag{2.2.10}$$

i.e. ψ^g is a symplectomorphism for every $g \in SL(2,\mathbb{R})$.

The Lie algebra of $SL(2,\mathbb{R})$ is

$$\mathfrak{sl}(2,\mathbb{R}) \;=\; \Big\{\, A \in \mathbb{R}^{2\times 2} : \operatorname{tr} A = 0 \,\Big\}, \tag{2.2.11}$$

the space of traceless 2×2 real matrices. These are matrices of the form

$$\begin{pmatrix} \alpha & \beta \\ \gamma & -\alpha \end{pmatrix} = \alpha \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + \beta \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + \gamma \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}. \tag{2.2.12}$$

The corresponding ODE of such a vector field is given by

$$\dot{q} = \alpha q + \beta p, \quad \dot{p} = \gamma q - \alpha p, \tag{2.2.13}$$

which itself belongs to the Hamiltonian given by

$$G = \alpha\langle q, p\rangle + \frac{\beta}{2}|p|^2 - \gamma|q|^2. \tag{2.2.14}$$

as a direct application of Hamilton's equations confirms.

So the integrals of the Kepler system are generated by the functions

$$r := |q|^2, s := |p|^2, u := \langle q, p\rangle. \tag{2.2.15}$$

This is easily verified also by hand

$$\dot{r} = 2\langle q, \dot{q}\rangle = 4ru - 4ur = 0, \tag{2.2.16}$$

$$\dot{s} = 2\langle p, \dot{p}\rangle = 4us - 4su = 0, \tag{2.2.17}$$

$$\dot{u} = \langle \dot{q}, p\rangle + \langle q, \dot{p}\rangle = \left(2rs - 2u^2\right) + \left(2u^2 - 2rs\right) = 0, \tag{2.2.18}$$

Periodicity of the flow of X_F. We now conclude that the Hamiltonian flow ϕ_F^t of $F = G_n^2$ is periodic. The Hamiltonian equations for F read

$$\dot{q} = \frac{\partial F}{\partial p} = 2|q|^2 p - 2\langle q, p\rangle q = 2(rp - uq), \qquad \dot{p} = -\frac{\partial F}{\partial q} = 2\langle q, p\rangle p - 2|p|^2 q = 2(up - sq). \tag{2.2.19}$$

Reduction to a linear ODE. Apparently, the coefficients of the ODE are integrals as derived in (2.2.15). In particular, $F = rs - u^2$ is constant along every trajectory, as expected.

Since r, s, u are constant along each trajectory, we may treat them as fixed parameters. The equations of motion (2.2.19) then reduce to the linear system

$$\begin{pmatrix} \dot{q} \\ \dot{p} \end{pmatrix} = 2M_0 \begin{pmatrix} q \\ p \end{pmatrix}, \qquad M_0 := \begin{pmatrix} -u & r \\ -s & u \end{pmatrix} \in \mathfrak{sl}(2, \mathbb{R}), \tag{2.2.20}$$

with unique solution $\phi_F^t(q_0, p_0) = e^{2tM_0}(q_0, p_0)$. Note that $e^{2tM_0} \in SL(2, \mathbb{R})$ for all t, since $\det\left(e^{2tM_0}\right) = e^{2t \operatorname{tr}(M_0)} = 1$, so the flow is precisely the restriction of the $SL(2, \mathbb{R})$-action ψ defined in (2.2.3) to the one-parameter subgroup $t \mapsto e^{2tM_0}$.

Since $\operatorname{tr}(M_0) = 0$, the characteristic polynomial of M_0 is $\lambda^2 + \det(M_0) = \lambda^2 + F$, yielding eigenvalues $\lambda_{1,2} = \pm i\sqrt{F}$ whenever $F > 0$. For a traceless 2×2 matrix A with eigenvalues $\pm i\mu$ ($\mu > 0$), the Cayley–Hamilton theorem gives $A^2 = -\mu^2 I$, and the matrix exponential takes the explicit form

$$e^{tA} = \cos(\mu t)\, I + \frac{\sin(\mu t)}{\mu}\, A. \tag{2.2.21}$$

Applying this to $A = 2M_0$ with $\mu = 2\sqrt{F}$, we obtain

$$e^{2tM_0} = \cos\!\Big(2\sqrt{F}\, t\Big)\, I + \frac{\sin\!\Big(2\sqrt{F}\, t\Big)}{\sqrt{F}}\, M_0, \tag{2.2.22}$$

which equals the identity precisely when $2\sqrt{F}\, t \in 2\pi\mathbb{Z}$. Hence ϕ_F^t is periodic with minimal period

$$T \;=\; \frac{\pi}{\sqrt{F}}. \tag{2.2.23}$$

If $F = 0$, the Cauchy–Schwarz inequality implies that q and p are linearly dependent. In this case one reads directly from (2.2.19) that $\dot{q} = \dot{p} = 0$, so every point of $\{F = 0\}$ is a fixed point of the flow, which is therefore trivially periodic.

In either case ϕ_F^t is periodic on all of $\mathbb{R}^{2n}$. Since each G_j^2 is simply G_n^2 restricted to the coordinate subspace $\mathbb{R}^{2j}$, the identical argument applies verbatim, and we conclude that every flow $\phi_{G_j^2}^t$ is periodic on its respective subspace, with period $\pi/\sqrt{G_j^2}$ on $\{G_j^2 > 0\}$.

The period of an orbit in the Kepler system is known to be

$$U(c) = 2\pi(-2c)^{-\frac{3}{2}} \text{ on the negative energy surface } \{H = c\} \text{ for } c < 0. \tag{2.2.24}$$

Normalization of the periods. Generally, a system with Hamiltonian $F = F(q, p)$ and periodic solutions on the energy surface $\{F = c\}$ where the period is $T = T(c)$, a diffeomorphism ϕ transforms the system in this way:

For the system defined by Hamiltonian $\phi(F)$, if $\phi'(c) \neq 0$, it holds that the solutions on the level set $\{\phi(F) = \phi(c)\}$ have period $(\frac{\partial \phi}{\partial c})^{-1} T(c)$.

We try to find a suitable ϕ, such that solutions have period 2π in the new system, so we solve the simple ODE and use (2.2.23)

$$\phi'(c)^{-1} \frac{\pi}{\sqrt{F}} = 2\pi \iff \phi'(c)^{-1} = 2\sqrt{F} \iff \phi'(c) = \frac{1}{2\sqrt{F}} \tag{2.2.25}$$

One reads off the solution (or use separation of variables) $\phi(c) = \sqrt{c} + C$. For simplicity, we choose $C = 0$.

Thus, the system given by the Hamiltonian $\sqrt{F} = G_n$ has solutions of period 2π, provided that $G_n > 0$.

We can normalize the periods of H by choosing $\tilde{\phi}(H) = (-2H)^{-\frac{1}{2}}$.

Hence, we set

$$G_{n+1} = (-2H)^{-\frac{1}{2}} \tag{2.2.26}$$

and by taking the square root of G_k^2, every $G_k \geq 0$ for all $k \geq 3$.

Geometry of the flows. Let $\phi_n^t = \exp(tX_{G_n})$ be the flow of X_{G_n} and we will interpret the points of $\mathbb{R}^{2n}$ as pairs of vectors (q,p) with $q,p \in \mathbb{R}^n$. Assume q and p linear independent firstly. Additionally, we define

$$E = \operatorname{span}\{q,p\} \tag{2.2.27}$$

and claim that $(q(t),p(t)) := \phi_n^t(q,p) \in E$ for all $t \in \mathbb{R}$ and that $(q(t),p(t))$ is obtained through rotation by angle t in E.

We show the claim by applying the canonical transformation $(q,p) \to (Rq, Rp)$ for a rotation R, that is chosen to yield $RE = \operatorname{span}\{e_1, e_2\}$.

On the subspace $RE \times RE$ the integral G_n becomes

$$G_n = \pm(q_1p_2 - q_2p_1), \tag{2.2.28}$$

which, up to the sign, defines the ODE

$$\dot{q_1} = -q_2, \qquad \dot{p_1} = -p_2 \tag{2.2.29}$$

$$\dot{q_2} = q_1 \qquad \dot{p_2} = p_1 \tag{2.2.30}$$

For which the assertion holds clearly.

If $2 \leq k < n$, we have the same notion on the projected plane $(\pi_k q, \pi_k p) \in \mathbb{R}^k$. Notice, that this pair $(\pi_k q, \pi_k p)$ can also be a line or point. In the case of a line, the orbit oscillates between two points and in the case of a point, we have a trivial orbit. Denote $E_k := \pi_k E = \operatorname{span}\{\pi_k q, \pi_k p\}$.

The last integral $G_{n+1} = (-2H)^{-\frac{1}{2}}$ is also called the *mean anomaly* in astronomy as it amounts to the factor, that rescales the initial orbits of the Kepler system to orbits of period 2π (2.2.26).

2.3. Independence of the differentials and the regular domain

So far we have produced n smooth functions

$$G_2,\ G_3,\ \ldots,\ G_n,\ G_{n+1} \tag{2.3.1}$$

on phase space, all in involution with H and with each other, and we have shown that the Hamiltonian flows generated by them are 2π-periodic on the open set where they are defined. To apply Theorem 1.1 we still need to specify the open subset of phase space on which the differentials $dG_2, \ldots, dG_n, dG_{n+1}$ are linearly independent. This is the content of the present subsection.

The candidate set. Define

$$\mathcal{D} \ := \ \Big\{ (q,p) \in \Big(\mathbb{R}^n \setminus \{0\}\Big) \times \mathbb{R}^n \ : \ 0 < G_2(q,p) < G_3(q,p) < \cdots < G_n(q,p) \Big\}. \tag{2.3.2}$$

Since $0 < G_{n+1} = (-2H)^{-\frac{1}{2}}$ in $\mathcal{D}$, we have $H < 0$ automatically.

Geometric meaning of the strict inequalities. The relation

$$G_k^2 - G_{k-1}^2 \ = \ \sum_{i<k} \Gamma_{ik}^2 \tag{2.3.3}$$

shows that $G_k > G_{k-1}$ holds precisely when at least one of the angular-momentum components Γ_{ik} with $i < k$ is non-zero. Geometrically, this means that the pair $(\pi_k q, \pi_k p) \in \mathbb{R}^{2k}$ spans a *genuinely two-dimensional* plane in $\mathbb{R}^k$ that is not already contained in $\mathbb{R}^{k-1}$. The condition $G_2 > 0$ excludes the degenerate “collision” orbits in which q and p are collinear (Cauchy–Schwarz equality), and the conditions $G_k > G_{k-1}$ exclude orbits whose plane lies inside a lower coordinate subspace. The full chain $0 < G_2 < \cdots < G_n$ thus describes the *generic position* of the orbit plane relative to the coordinate flag $\mathbb{R}^2 \subset \mathbb{R}^3 \subset \cdots \subset \mathbb{R}^n$.

Proposition 2.2. *On $\mathcal{D}$ the n differentials*

$$dG_2,\ dG_3,\ \ldots,\ dG_n,\ dG_{n+1} \tag{2.3.4}$$

are linearly independent at every point.

The proof uses two structural observations and then a triangular-rank induction; We record the key observations here since they also shed geometric light on the regular domain $\mathcal{D}$.

Observation 1: support. For $k \in \{2, \ldots, n\}$ the function G_k depends only on the first $2k$ coordinates $(q_1, \ldots, q_k, p_1, \ldots, p_k)$. Consequently

$$\frac{\partial G_k}{\partial q_l} = \frac{\partial G_k}{\partial p_l} = 0, \qquad l > k. \tag{2.3.5}$$

Observation 2: action of the partial derivatives at index k. From $G_k^2 = \sum_{i<j\le k} \Gamma_{ij}^2$ we compute

$$\frac{\partial G_k^2}{\partial p_l} = 2 \sum_{i<j\le k} \Gamma_{ij}\Big(q_i \delta_{jl} - q_j \delta_{il}\Big). \tag{2.3.6}$$

Specialising to $l = k$ and using that $i < j \le k$ excludes $i = k$, only the δ_{jk} term survives:

$$\frac{\partial G_k^2}{\partial p_k} = 2\sum_{i<k} \Gamma_{ik}\, q_i. \tag{2.3.7}$$

An identical computation gives

$$\frac{\partial G_k^2}{\partial q_k} = -2\sum_{i<k} \Gamma_{ik}\, p_i. \tag{2.3.8}$$

Using $dG_k = (2G_k)^{-1}\, dG_k^2$, we conclude

$$\frac{\partial G_k}{\partial p_k} = \frac{1}{G_k} \sum_{i<k} \Gamma_{ik}\, q_i, \qquad \frac{\partial G_k}{\partial q_k} = -\frac{1}{G_k} \sum_{i<k} \Gamma_{ik}\, p_i. \tag{2.3.9}$$

Observation 3: G_{n+1} and H. From $G_{n+1} = (-2H)^{-1/2}$ we have $dG_{n+1} = G_{n+1}^3\, dH$, hence

$$\frac{\partial G_{n+1}}{\partial p_l} = G_{n+1}^3\, p_l, \qquad \frac{\partial G_{n+1}}{\partial q_l} = G_{n+1}^3\, \frac{q_l}{|q|^3}. \tag{2.3.10}$$

In particular, $\partial G_{n+1}/\partial p_l$ does *not* vanish for any l unless $p_l = 0$.

Remark 2.3. *Since $dG_{n+1} = G_{n+1}^3\, dH$ and $G_{n+1} > 0$ on $\mathcal{D}$, replacing G_{n+1} by H in (2.3.4) gives an equivalent system of differentials. This is convenient when we want H to appear explicitly among the integrals (as in the statement of Theorem 1.1); we shall use the two formulations interchangeably.*

Remark 2.4. *Let $F = (G_2, \ldots, G_n, G_{n+1})\colon \mathcal{D} \to \mathbb{R}^n$. Proposition 2.2 says exactly that every $c = (c_2, \ldots, c_n, c_{n+1}) \in F(\mathcal{D})$ is a regular value of F. The fibre $F^{-1}(c)$ – the joint level set – is therefore an embedded n-dimensional submanifold of $\mathcal{D}$. We will see in the next subsection that, for c corresponding to a bounded Kepler orbit, this fibre is also compact and connected,* *and hence diffeomorphic to $\mathbb{T}^n$ by Theorem 1.1.*

2.4. Construction of the Delaunay variables

We are now in a position to apply Theorem 1.1 to the Kepler problem. Before summarising the hypotheses we make the construction fully explicit by exhibiting a concrete transversal section to the torus fibres, following Moser and Zehnder [2005, Chapter 3, §3.4].

The transversal section S

In order to turn the abstract angle coordinates of Theorem 1.1 into concrete functions on phase space, we need a smooth hypersurface $S \subset \mathcal{D}$ that meets each torus fibre $N_c = F^{-1}(c)$ transversally in exactly one point. Such a section simultaneously fixes the "starting phase" for each of the n periodic flows $\phi^t_{G_k}$.

Following Moser–Zehnder, define

$$S \;:=\; \Big\{ (q,p) \in \Big(\mathbb{R}^n \setminus \{0\}\Big) \times \mathbb{R}^n \;:\; q_1 > 0,\;\; q_j = 0 \text{ for } j \geq 2,\;\; p_1 = 0,\;\; p_j > 0 \text{ for } j \geq 2 \Big\}. \tag{2.4.1}$$

Geometrically, a point of S has its position vector along the positive q_1-semiaxis (perihelion direction), its velocity purely transverse to this axis ($p_1 = 0$), and each successive transverse component $p_j > 0$. These conditions pin down the position along the orbit as well as the relative orientation of each coordinate two-plane.

To verify that S parameterises the action values correctly, we restrict the equations $G_{k+1}(q,p) = y_k$ to S. Using $q_j = 0$ for $j \geq 2$ and $p_1 = 0$, a direct computation gives $\Gamma_{ij} = q_i p_j$ for $j \geq 2$ and $\Gamma_{ij} = 0$ otherwise, so

$$G_{k+1}^2 \;=\; \sum_{j=2}^{k+1} \Gamma_{1j}^2 \;=\; q_1^2 \sum_{j=2}^{k+1} p_j^2 \;=\; q_1^2\, |\pi_{k+1} p|^2,$$

where π_{k+1} denotes the projection onto the first $k+1$ coordinates. The system $G_{k+1} = y_k$ therefore reduces on S to the explicit triangular system

$$\begin{cases} q_1 p_2 \;=\; y_1, & \\ |q_1|\,|\pi_{k+1} p| \;=\; y_k, & k = 2, \ldots, n-1, \\ -|p|^2 + 2|q_1|^{-1} \;=\; y_n^{-2}, & \end{cases} \tag{2.4.2}$$

where the last equation is equivalent to $H = -\frac{1}{2} y_n^{-2}$, i.e. $G_{n+1} = (-2H)^{-1/2} = y_n$.

Lemma 2.5 (Moser and Zehnder [2005, Lemma 3.12]). *Let*

$$\Omega \ := \ \Big\{ y \in \mathbb{R}^n : 0 < |y_1| < y_2 < \cdots < y_n \Big\}. \tag{2.4.3}$$

For each $y \in \Omega$ the system $G_{k+1}(q,p) = y_k$, $k = 1, \ldots, n$, has a unique solution $(q,p) = \lambda(y)$ in S. The map $\lambda\colon \Omega \to S \cap \mathcal{D}$ is a diffeomorphism.

Proof. We solve system (2.4.2) from the bottom up.

Step 1: Determine $|p|$ and q_1. The equation for $k = n-1$ gives $|q_1||p| = y_{n-1}$ (since $|\pi_n p| = |p|$ on S as $p_1 = 0$). Substituting $|q_1| = y_{n-1}/|p|$ into the energy equation:

$$-|p|^2 + \frac{2|p|}{y_{n-1}} \ = \ y_n^{-2}, \qquad \text{i.e.,} \qquad \Big(|p| - y_{n-1}^{-1}\Big)^2 \ = \ y_{n-1}^{-2} - y_n^{-2} \ > \ 0,$$

where the right-hand side is positive because $y_{n-1} < y_n$. Of the two positive roots, only the one satisfying $|p|\, y_{n-1} > 1$ lies in $\mathcal{D}$ (it corresponds to bounded, non-degenerate orbits). This uniquely determines $|p|$ and hence $q_1 = y_{n-1}/|p| > 0$.

Step 2: Determine p_k for $k = 2, \ldots, n$ inductively. From $|q_1|\,|\pi_k p| = y_{k-1}$ we read off $|\pi_k p| = y_{k-1}/q_1$. Since $p_1 = 0$,

$$p_k^2 \ = \ |\pi_k p|^2 - |\pi_{k-1} p|^2 \ = \ \frac{y_{k-1}^2 - y_{k-2}^2}{q_1^2} \ > \ 0,$$

using $y_{k-2} < y_{k-1}$. The sign condition $p_k > 0$ on S selects the positive square root. For $k = 2$ one uses $q_1 p_2 = y_1$ directly.

Smoothness and global injectivity. The invertibility of the map $(q,p) \mapsto (G_2, \ldots, G_{n+1})$ restricted to S follows from the implicit function theorem: in the variables $\alpha_j := q_1 p_{j+1}$ $(j = 1, \ldots, n-1)$ and $\alpha_n := q_1$, the Jacobian $(\partial G_{k+1}/\partial \alpha_j)$ is lower triangular with non-zero diagonal entries on $\mathcal{D}$, so it is invertible. Global injectivity is clear from the uniqueness established above. Q.E.D.

Lemma 2.5 shows that $S \cap \mathcal{D}$ is diffeomorphic to Ω via the action values alone. It also confirms that the construction of Theorem 1.1 applies: each torus N_c meets S in precisely one point. The hypotheses are summarised in the following statement, that is the exact projection of the general statement by Arnold and Jost onto the Kepler System.

We now switch to the classical notations of the action-angle variables of the Kepler System $\Big(\ell_2, \ldots, \ell_n, \ell_{n+1};\, L_2, \ldots, L_n, L_{n+1}\Big)$.

Theorem 2.6 (Arnold–Jost for the Kepler problem)**.** *Let $(\mathcal{D}, \omega_0|_{\mathcal{D}})$ be the regular domain of the Kepler problem defined in (2.3.2), and let*

$$F \;=\; (G_2,\, G_3,\, \ldots,\, G_n,\, G_{n+1})\colon \mathcal{D} \to \mathbb{R}^n. \tag{2.4.4}$$

Then:

(i) the components of F are in pairwise involution, $\{F_i, F_j\} = 0$ for all i, j (Lemma 2.1);

(ii) their differentials $dF_1, \ldots, dF_n$ are linearly independent at every point of $\mathcal{D}$ (Proposition 2.2);

(iii) $\{H, F_i\} = 0$ for all i (involution of H with each Γ_{ij} together with the Leibniz rule);

(iv) the flows of $X_{F_1}, \ldots, X_{F_n}$ are complete and 2π-periodic on $\mathcal{D}$ (Section 2.2).

Hence, for every regular value $c = (c_2, \ldots, c_n, c_{n+1}) \in F(\mathcal{D})$ such that the fibre $N_c := F^{-1}(c)$ is compact and connected, there exists an open neighbourhood $U(N_c) \subset \mathcal{D}$ and a diffeomorphism

$$\Psi\colon \mathbb{T}^n \times D_1 \;\longrightarrow\; U(N_c), \tag{2.4.5}$$

such that $\Psi^*\omega_0 = \sum_{k=1}^n dL_k \wedge d\ell_k$, where the new coordinates

$$\Big(\ell_2, \ldots, \ell_n, \ell_{n+1};\, L_2, \ldots, L_n, L_{n+1}\Big) \;\in\; \mathbb{T}^n \times D_1 \tag{2.4.6}$$

are the Delaunay variables*. Here the actions are nothing but the integrals themselves,*

$$L_k \;=\; G_k, \qquad k = 2, \ldots, n+1, \tag{2.4.7}$$

and the angles ℓ_k are determined modulo 2π by the requirement that the time-1 flow of X_{L_k} act as the rotation $\ell_k \mapsto \ell_k + 1$.

Proof. Items (i)–(iv) are explicit restatements of results in Sections 2.1 and 2.2 and Proposition 2.2. The existence of the action-angle chart is Theorem 1.1; identifying $L_k = G_k$ is Step 5 in the proof of Theorem 1.4, applied to the family $F_k = G_k$. The normalisation $\ell_k \sim \ell_k + 1 \bmod 2\pi$ is what was secured by the period-normalisation step $\phi(c) = \sqrt{c}$ for

$k = 2, \ldots, n$ and $\tilde{\phi}(H) = (-2H)^{-1/2}$ for $k = n+1$: indeed, after this normalisation each flow $\phi^t_{L_k}$ is 2π-periodic, so its orbit is canonically parametrised by an angle $\ell_k \in \mathbb{R}/2\pi\mathbb{Z}$. Q.E.D.

Compactness and connectedness of the fibre. The fibre $N_c = F^{-1}(c)$ is compact: bounded energy ($H = -\frac{1}{2}L_{n+1}^{-2} < 0$) confines $|q|$ to a bounded interval (Kepler's equation $H = \frac{1}{2}|p|^2 - |q|^{-1} = c$ together with $|q|^{-1} \geq -c$ gives $|q| \leq -c^{-1}$), and the angular-momentum bound $|q|\,|p| \geq L_n$ combined with $\frac{1}{2}|p|^2 \leq c + |q|^{-1}$ keeps $|p|$ bounded as well. Connectedness follows from the explicit description of the flow as the $\mathbb{R}^n$-orbit of any single point of N_c (Section 1.3, applied with $\Gamma \subset \mathbb{R}^n$ of full rank).

2.5. Interpretation of the Delaunay variables

The Delaunay variables are of direct physical significance in the case $n = 3$, so the Kepler problem in three-dimensional space. These variables are usually introduced via separation of variables in polar coordinates for the Hamilton–Jacobi equation [Arnold, 1989]; the present approach through the Arnold–Jost theorem is more transparent geometrically, even if probably a bit longer.

The action variables and orbital elements

In three dimensions the phase space is $\mathbb{R}^6$ and the theorem produces three action variables L_2, L_3, L_4 (using the indexing $L_k = G_k,\ k = 2, 3, 4$) with conjugate angles ℓ_2, ℓ_3, ℓ_4. The actions admit classical interpretations in terms of orbital elements.

Here L_2 decodes the q_3-component of the angular momentum vector $q \wedge p$ (the component perpendicular to the reference plane $\pi_2\mathbb{R}^3 = \{q_3 = 0\}$), while L_3 is the magnitude of the total angular momentum. Writing j for the inclination of the orbit plane E against $\pi_2\mathbb{R}^3$, one in fact has

$$L_2 = L_3 \cos j, \tag{2.5.1}$$

so the condition $0 < L_2 < L_3$ from (2.3.2) is precisely $0 < \cos j < 1$, i.e. the orbit plane is neither horizontal ($j \neq 0$) nor vertical ($j \neq \pi/2$).

For the energy action L_4, the standard theory of Keplerian ellipses gives

$$L_4 = \sqrt{a}, \tag{2.5.2}$$

where $a > 0$ is the semi-major axis of the elliptic orbit. The angular momentum is related to a and the eccentricity ε by

$$L_3 = G_3 = L_4\sqrt{1-\varepsilon^2} = \sqrt{a(1-\varepsilon^2)}, \tag{2.5.3}$$

which is the semi-latus rectum of the ellipse. The condition $L_3 < L_4$ is therefore equivalent to $\varepsilon < 1$, i.e. the orbit is a (proper) ellipse rather than a parabola or hyperbola – consistent with the requirement $H < 0$ already imposed in $\mathcal{D}$.

The angle variables

The action variables L_k have been interpreted geometrically above. The angle variables ℓ_k require no separate construction: they are uniquely determined – up to the choice of reference section S from Lemma 2.5 – by the two structural conditions of Theorem 1.1. The first is *canonicity*: the coordinate change satisfies $\Psi^*\omega = \sum_k dL_k \wedge d\ell_k$. The second is that the *integrals are independent of the angles*: $F_j \circ \Psi = L_j$ for all j, so the L_k are functions of the action coordinates alone. Given the action variables and the reference section S, these two conditions together uniquely pin down the ℓ_k; the angles emerge from the symplectic structure rather than from an explicit formula. Next follows their geometric meaning:

- ℓ_4 (conjugate to $L_4 = G_4$) is the *mean anomaly*: it parametrises position along the orbit at a rate proportional to time, and increases by 2π in one orbital period.
- ℓ_3 (conjugate to $L_3 = G_3$) is the *argument of perihelion*: the angle in the orbit plane from the ascending node to the perihelion direction.
- ℓ_2 (conjugate to $L_2 = G_2$) is the *longitude of the ascending node*: the angle in the reference plane $\pi_2\mathbb{R}^3$ to the line where the orbit plane crosses it.

In these coordinates the Kepler Hamiltonian, expressed as a function of the action variables via (2.5.2), takes the completely separated form

$$H = -\frac{1}{2L_4^2}, \tag{2.5.4}$$

depending only on L_4. This is the hallmark of complete integrability: the Hamiltonian reduces to a function of actions alone, and the equations of motion in Delaunay coordinates become simply $\dot{L}_k = 0$, $\dot{\ell}_k = \partial H/\partial L_k$, with $\partial H/\partial L_4 = L_4^{-3}$ the mean motion and $\partial H/\partial L_2 = \partial H/\partial L_3 = 0$.

A. Fundamentals of Symplectic Geometry and Hamiltonian Mechanics

Before turning to the theorem of Arnold and Jost and its applications, we collect the basic definitions and structures from symplectic geometry that we rely on throughout. The treatment is intentionally concise: each concept is introduced in exactly the form needed for what follows. For a comprehensive reference, the reader is directed to da Silva [2001]; a broader treatment in the context of topology is given in McDuff and Salamon [2017].

Definition 1. *(Linear symplectic structure)* Let V be a finite-dimensional real vector space. A bilinear, skew-symmetric map

$$\omega\colon V \times V \to \mathbb{R} \tag{A.0.1}$$

is called a *symplectic form* on V if the induced linear map $\tilde{\omega}\colon V \to V^*$ defined by

$$\tilde{\omega}(v) = \omega(v, \cdot\,) \tag{A.0.2}$$

is an isomorphism. This second condition is called *non-degeneracy*, and the pair (V, ω) is then called a *symplectic vector space*.

This motivates the definition of a symplectic manifold through:

Definition 2. *(Symplectic manifold)* A differential 2-form ω on a smooth manifold M is called *symplectic* if

(i) $\omega_p\colon T_pM \times T_pM \to \mathbb{R}$ is a symplectic form for every $p \in M$ (non-degeneracy), and

(ii) $d\omega = 0$ (closedness).

The pair (M, ω) is then called a *symplectic manifold*.

Non-degeneracy forces $\dim M = 2n$ to be even. A central result of symplectic geometry, whose statement will be used repeatedly in what follows, is the theorem of Darboux.

Theorem A.1. *(Darboux) For any symplectic manifold (M, ω) and any point $p \in M$, there exist local coordinates $(q_1, \ldots, q_n, p_1, \ldots, p_n)$ in a neighbourhood of p such that*

$$\omega = \sum_{i=1}^{n} dp_i \wedge dq_i. \tag{A.0.3}$$

The theorem of Darboux tells us that all symplectic manifolds of the same dimension look locally alike: there are no local symplectic invariants. The canonical local model is therefore

$(\mathbb{R}^{2n}, \omega_0)$, where

$$\omega_0 = \sum_{i=1}^{n} dp_i \wedge dq_i. \tag{A.0.4}$$

A.1. Hamiltonian mechanics

Throughout this subsection, let (M, ω) denote a symplectic manifold.

Proposition A.2. *For every smooth function $H\colon M \to \mathbb{R}$ there exists a unique vector field $X_H \in \Gamma(TM)$ satisfying*

$$\iota_{X_H}\omega \;=\; \omega(X_H, \cdot\,) \;=\; -\,dH. \tag{A.1.1}$$

Since $-dH$ is a differential 1-form, existence and uniqueness of X_H follow pointwise from the non-degeneracy of ω_p (Definition 1).

Definition 3. In the setting above, X_H is called the *Hamiltonian vector field* with *Hamiltonian function H*.

Definition 4. *(Symplectomorphism)* A diffeomorphism $\varphi\colon (M, \omega) \to (M', \omega')$ between symplectic manifolds is called a *symplectomorphism* if $\varphi^*\omega' = \omega$.

In what follows we will repeatedly encounter the Kepler system, which lives in Euclidean space. It is therefore worth recording the explicit form of Hamiltonian mechanics on $\mathbb{R}^{2n}$. Consider $\mathbb{R}^{2n}$ with coordinates $(q_1, \dots, q_n, p_1, \dots, p_n)$ and the standard symplectic form ω_0 from (A.0.4).

Definition 5. *(Hamilton's equations)* A curve $\rho(t) = (q(t), p(t))$ is an integral curve of X_H if and only if

$$\begin{cases} \dfrac{dq_i}{dt}(t) \;=\; \dfrac{\partial H}{\partial p_i}\big(\rho(t)\big) \\ \dfrac{dp_i}{dt}(t) \;=\; -\dfrac{\partial H}{\partial q_i}\big(\rho(t)\big) \end{cases} \qquad i = 1, \dots, n. \quad \textbf{(Hamilton's equations)} \tag{A.1.2}$$

These equations follow directly from the defining relation $\iota_{X_H}\omega_0 = -dH$. Indeed, setting

$$X_H = \sum_{i=1}^{n} \left(\frac{\partial H}{\partial p_i} \frac{\partial}{\partial q_i} - \frac{\partial H}{\partial q_i} \frac{\partial}{\partial p_i} \right), \tag{A.1.3}$$

a direct computation confirms $\iota_{X_H}\omega_0 = -dH$. Writing the integral-curve condition $\dot\rho(t) = X_H(\rho(t))$ component-wise,

$$\sum_{i=1}^{n}\left(\dot q_i(t)\,\frac{\partial}{\partial q_i} + \dot p_i(t)\,\frac{\partial}{\partial p_i}\right) = \sum_{i=1}^{n}\left(\frac{\partial H}{\partial p_i}\,\frac{\partial}{\partial q_i} - \frac{\partial H}{\partial q_i}\,\frac{\partial}{\partial p_i}\right)\Bigg|_{\rho(t)}, \tag{A.1.4}$$

and comparing coefficients recovers Hamilton's equations.

Definition 6. *(Poisson bracket)* For smooth functions $f, g\colon M \to \mathbb{R}$ on a symplectic manifold (M,ω), the *Poisson bracket* is defined by

$$\{f,\, g\} \;:=\; \omega(X_f, X_g) = df(X_g). \tag{A.1.5}$$

In local Darboux coordinates (q_i, p_i) this reads

$$\{f,\, g\} = \sum_{i=1}^{n}\left(\frac{\partial f}{\partial q_i}\frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i}\frac{\partial g}{\partial q_i}\right). \tag{A.1.6}$$

Remark A.3. *The time derivative of a smooth observable $f\colon M \to \mathbb{R}$ along a Hamiltonian flow satisfies*

$$\frac{d}{dt}f(\rho(t)) = \{f,\, H\}(\rho(t))\,. \tag{A.1.7}$$

In particular, f is a first integral *(constant of motion) of H if and only if $\{f,\, H\} = 0$. Setting $f = H$ gives $\{H,H\} = 0$, so the Hamiltonian itself is always a first integral: energy is conserved along Hamiltonian flows. Two functions satisfying $\{f, g\} = 0$ are said to be* in involution*; this notion is central to the formulation of the Arnold–Jost theorem.*

B. Computational Details

This appendix collects the more computational proofs from Section 2, which were deferred in order to keep the main exposition focused on the key ideas.

B.1. The Poisson bracket of angular-momentum components

We verify formula (2.1.6), which states

$$\{\Gamma_{ij}, \Gamma_{\alpha\beta}\} = \Gamma_{j\alpha}\delta_{i\beta} - \Gamma_{i\alpha}\delta_{j\beta} - \Gamma_{j\beta}\delta_{i\alpha} + \Gamma_{i\beta}\delta_{j\alpha}.$$

Recall the Poisson bracket on $\mathbb{R}^{2n}$,

$$\{F, G\} = \sum_k \left(\frac{\partial F}{\partial p_k} \frac{\partial G}{\partial q_k} - \frac{\partial F}{\partial q_k} \frac{\partial G}{\partial p_k} \right),$$

and $\Gamma_{ij} = q_i p_j - q_j p_i$. We compute

$$\{\Gamma_{ij}, \Gamma_{\alpha\beta}\} = \sum_k \left(\frac{\partial \Gamma_{ij}}{\partial p_k} \frac{\partial \Gamma_{\alpha\beta}}{\partial q_k} - \frac{\partial \Gamma_{ij}}{\partial q_k} \frac{\partial \Gamma_{\alpha\beta}}{\partial p_k} \right). \tag{B.1.1}$$

The partial derivatives of $\Gamma_{ij} = q_i p_j - q_j p_i$ are

$$\frac{\partial \Gamma_{ij}}{\partial q_k} = \delta_{ik} p_j - \delta_{jk} p_i, \qquad \frac{\partial \Gamma_{ij}}{\partial p_k} = q_i \delta_{jk} - q_j \delta_{ik},$$

and analogously for $\Gamma_{\alpha\beta}$. Substituting and summing over k,

$$\begin{aligned}
\sum_k \frac{\partial \Gamma_{ij}}{\partial p_k} \frac{\partial \Gamma_{\alpha\beta}}{\partial q_k} &= \sum_k (q_i \delta_{jk} - q_j \delta_{ik})(\delta_{\alpha k} p_\beta - \delta_{\beta k} p_\alpha) \\
&= q_i p_\beta \, \delta_{j\alpha} - q_i p_\alpha \, \delta_{j\beta} - q_j p_\beta \, \delta_{i\alpha} + q_j p_\alpha \, \delta_{i\beta}, \\
\sum_k \frac{\partial \Gamma_{ij}}{\partial q_k} \frac{\partial \Gamma_{\alpha\beta}}{\partial p_k} &= \sum_k (\delta_{ik} p_j - \delta_{jk} p_i)(q_\alpha \delta_{\beta k} - q_\beta \delta_{\alpha k}) \\
&= p_j q_\alpha \, \delta_{i\beta} - p_j q_\beta \, \delta_{i\alpha} - p_i q_\alpha \, \delta_{j\beta} + p_i q_\beta \, \delta_{j\alpha}.
\end{aligned}$$

Taking the difference and grouping by Kronecker symbol,

$$\begin{aligned}
\{\Gamma_{ij}, \Gamma_{\alpha\beta}\} = {} & (q_i p_\beta - p_i q_\beta) \, \delta_{j\alpha} - (q_i p_\alpha - p_i q_\alpha) \, \delta_{j\beta} \\
& - (q_j p_\beta - p_j q_\beta) \, \delta_{i\alpha} + (q_j p_\alpha - p_j q_\alpha) \, \delta_{i\beta}.
\end{aligned}$$

Each factor is itself a Γ, yielding

$$\{\Gamma_{ij}, \Gamma_{\alpha\beta}\} = \Gamma_{j\alpha} \delta_{i\beta} - \Gamma_{i\alpha} \delta_{j\beta} - \Gamma_{j\beta} \delta_{i\alpha} + \Gamma_{i\beta} \delta_{j\alpha},$$

which is (2.1.6). Q.E.D.

B.2. The G_k^2 are in involution

We prove Lemma 2.1: $\{G_k^2, G_\ell^2\} = 0$ for all k, ℓ.

Proof. We first show that $\{G_k^2, \Gamma_{\alpha\beta}\} = 0$ whenever $\alpha, \beta \leq k$, and then apply Leibniz a second time to conclude.

Step 1: Leibniz rule. Since $G_k^2 = \sum_{1\le i<j\le k} \Gamma_{ij}^2$, the Leibniz rule $\{fg,h\} = f\{g,h\} + g\{f,h\}$ gives

$$\{G_k^2, \Gamma_{\alpha\beta}\} = \sum_{1\le i<j\le k} \{\Gamma_{ij}^2, \Gamma_{\alpha\beta}\} = \sum_{1\le i<j\le k} 2\Gamma_{ij}\{\Gamma_{ij}, \Gamma_{\alpha\beta}\}. \tag{B.2.1}$$

Step 2: Extending the sum to all i, j. The antisymmetry $\Gamma_{ij} = -\Gamma_{ji}$ implies

$$\Gamma_{ji}\{\Gamma_{ji}, \Gamma_{\alpha\beta}\} = (-\Gamma_{ij})\Big(-\{\Gamma_{ij}, \Gamma_{\alpha\beta}\}\Big) = \Gamma_{ij}\{\Gamma_{ij}, \Gamma_{\alpha\beta}\},$$

so every pair (i,j) with $i<j$ contributes the same amount as its transpose (j,i). Diagonal terms vanish since $\Gamma_{ii} = 0$, and we can therefore extend the summation to all of $\{1,\ldots,k\}^2$:

$$\{G_k^2, \Gamma_{\alpha\beta}\} = \sum_{i,j=1}^{k} \Gamma_{ij}\{\Gamma_{ij}, \Gamma_{\alpha\beta}\}.$$

Step 3: Inserting (2.1.6). Substituting the bracket formula (2.1.6):

$$\{G_k^2, \Gamma_{\alpha\beta}\} = \sum_{i,j=1}^{k} \Gamma_{ij}(\Gamma_{j\alpha}\delta_{i\beta} - \Gamma_{i\alpha}\delta_{j\beta} - \Gamma_{j\beta}\delta_{i\alpha} + \Gamma_{i\beta}\delta_{j\alpha})\,.$$

Step 4: Symmetrisation via $\Gamma_{ij} = -\Gamma_{ji}$. Relabelling $i \leftrightarrow j$ in the first summand,

$$\sum_{i,j=1}^{k} \Gamma_{ij}\,\Gamma_{j\alpha}\delta_{i\beta} \overset{i\leftrightarrow j}{=} \sum_{i,j=1}^{k} \Gamma_{ji}\,\Gamma_{i\alpha}\delta_{j\beta} \;=\; -\sum_{i,j=1}^{k} \Gamma_{ij}\,\Gamma_{i\alpha}\delta_{j\beta}.$$

Hence the first summand equals minus the second, and the same argument applies to the third and fourth. Collecting the two pairs gives

$$\{G_k^2, \Gamma_{\alpha\beta}\} = 2\sum_{i,j=1}^{k} \Gamma_{ij}(\Gamma_{i\beta}\,\delta_{j\alpha} - \Gamma_{i\alpha}\,\delta_{j\beta})\,. \tag{B.2.2}$$

Step 5: Evaluating for $\alpha, \beta \le k$. Both α, β lie within the summation range, so the Kronecker symbols fire at $j = \alpha$ and $j = \beta$ respectively:

$$\{G_k^2, \Gamma_{\alpha\beta}\} = 2\sum_{i=1}^{k}(\Gamma_{i\alpha}\Gamma_{i\beta} - \Gamma_{i\beta}\Gamma_{i\alpha}) = 0, \tag{B.2.3}$$

since the Γ_{ij} are smooth real-valued functions and hence commute as multiplication operators.

Conclusion. Applying Leibniz once more to $G_\ell^2 = \sum_{1\le\alpha<\beta\le\ell}\Gamma_{\alpha\beta}^2$ and using (B.2.3),

$$\{G_k^2, G_\ell^2\} = 2\sum_{1\le\alpha<\beta\le\ell}\{G_k^2, \Gamma_{\alpha\beta}\}\,\Gamma_{\alpha\beta} = 0, \tag{B.2.4}$$

where for $\ell \le k$ every α, β satisfies $\alpha, \beta \le \ell \le k$, so (B.2.3) applies. Q.E.D.

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Name:	Carl Walter
Matrikel-Nr.:	4263232

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